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Existence of positive solutions for $2n$th-order singular superlinear boundary value problems ☆

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ABSTRACT

This paper investigates the existence of positive solutions for $2n$th-order ($n > 1$) singular superlinear boundary value problems. A necessary and sufficient condition for the existence of $C^{2n-2}[0, 1]$ as well as $C^{2n-1}[0, 1]$ positive solutions is given by constructing a special cone and with the e-Norm.

1. Introduction

Singular ordinary differential equations arise in the fields of gas dynamics, Newtonian fluid mechanics, boundary layer and so on. The theory of singular boundary value problems has become an important area of investigation in recent years (see [1–4] and references therein). Consider the $2n$th-order ordinary differential equation

$$(-1)^n x^{(2n)}(t) = f\left(t, x(t), -x''(t), \ldots, (-1)^i x^{(2i)}(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t)\right), \quad 0 < t < 1, \tag{1.1}$$

with the boundary conditions

$$x^{(2i)}(0) = x^{(2i)}(1) = 0 \quad i = 0, 1, 2, \ldots, n-1, \tag{1.2}$$

where $n > 1$ is an integer, and f satisfies the following hypothesis:
(**H**) $f \in C((0, 1) \times (0, \infty)^n, [0, \infty))$, and there exist constants λ_i, μ_i ($0 < \lambda_i \le \mu_i, i = 1, 2, \ldots, n, \sum_{i=1}^{n} \lambda_i > 1$) such that for $t \in (0, 1), x_i \in (0, \infty)$,

$$\begin{aligned} &c^{\mu_i} f(t, x_1, x_2, \ldots, x_i, \ldots, x_n) \le f(t, x_1, x_2, \ldots, cx_i, \ldots, x_n) \\ &\le c^{\lambda_i} f(t, x_1, x_2, \ldots, x_i, \ldots, x_n), \quad \text{if } 0 < c \le 1, i = 1, 2, \ldots, n. \end{aligned} \tag{1.3}$$

Remark 1. (1.3) implies

$$\begin{aligned} &c^{\lambda_i} f(t, x_1, x_2, \ldots, x_i, \ldots, x_n) \le f(t, x_1, x_2, \ldots, cx_i, \ldots, x_n) \\ &\le c^{\mu_i} f(t, x_1, x_2, \ldots, x_i, \ldots, x_n), \quad \text{if } c \ge 1, i = 1, 2, \ldots, n, \end{aligned} \tag{1.4}$$

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and

$$f(t, x_1, x_2, \dots, x_n) \le f(t, y_1, y_2, \dots, y_n), \quad \text{if } 0 < x_i \le y_i, \ i = 1, 2, \dots, n. \tag{1.5}$$

Typical functions that satisfy the above superlinear hypothesis are those taking the form $f(t, x_1, x_2, \dots, x_n) = \sum_{i=1}^{m} p_i(t) x_1^{l_{i1}} x_2^{l_{i2}} \dots x_n^{l_{in}}$; here $p_j(t) \in C(0, 1)$, $p_j(t) > 0$ on $(0, 1)$, $l_{jk} > 0$, $\sum_{k=1}^{n} l_{jk} > 1$, $j = 1, 2, \dots, m$, $k = 1, 2, \dots, n$.

By singularity we mean that the function $f(t, x_1, x_2, , \dots, x_n)$ in (1.1) is allowed to be unbounded at $t = 0$ and/or $t = 1$. A function $x(t) \in C^{2n-2}[0, 1] \cap C^{2n}(0, 1)$ is called a $C^{2n-2}[0, 1]$(positive) solution of (1.1) and (1.2) if it satisfies (1.1) and (1.2) ($(-1)^i x^{(2i)}(t) > 0, i = 1, 2, \dots, n-1$ for $t \in (0, 1)$). A $C^{2n-2}[0, 1]$ (positive) solution of (1.1) and (1.2) is called a $C^{2n-1}[0, 1]$ (positive) solution if $x^{(2n-1)}(0^+)$ and $x^{(2n-1)}(1^-)$ both exist ($(-1)^i x^{(2i)}(t) > 0, i = 1, 2, \dots, n-1$ for $t \in (0, 1)$).

For the special case $n = 2$, the function $f \in C([0, 1] \times R \times R, R)$ in (1.1), i.e., f is continuous, problem (1.1) and (1.2) is nonsingular, the existence and uniqueness of solutions of (1.1) and (1.2) have been studied by papers [5–12]. A sufficient condition for the existence of solutions of the singular problem (1.1) and (1.2) was given by O'Regan in [10] with a topological transversal theorem.

For the general case of $n > 2$, in which the problem (1.1) and (1.2) has received attention in the literature, we refer the reader to [13–18]. For instance, in the sublinear case, Wei [16] gave some necessary and sufficient conditions for the existence results of a class of singular boundary value problems by the upper and lower solutions method. In the superlinear case ($\lambda_i > 1$ ($i = 1, 2, \dots, n-1$) and $0 < \lambda_n \le \mu_n < 1$), Shi and Chen [14] obtained some necessary and sufficient conditions for the existence of $C^{2n-2}[0, 1]$ as well as $C^{2n-1}[0, 1]$ positive solutions by means of the fixed point theorems on cones.

In this paper, we shall study the existence of positive solutions for $2n$th-order singular superlinear boundary value problems (1.1) and (1.2) ($\sum_{i=1}^{n} \lambda_i > 1$). A necessary and sufficient condition for the existence of $C^{2n-2}[0, 1]$ as well as $C^{2n-1}[0, 1]$ positive solutions is given by constructing a special cone and with e-Norm. The conclusions obtained improve the main results in [14]. Our nonlinearity $f(t, x_1, x_2, \dots, x_n)$ may be singular at $t = 0$ and/or $t = 1$.

2. Main results

For convenience, we set

$$F_1(t) = f(t, t(1-t), t(1-t), \dots, t(1-t), 1),$$
$$F_2(t) = f(t, t(1-t), t(1-t), \dots, t(1-t), t(1-t)).$$

Our main results are the following theorems.

Theorem 2.1. *Suppose* (**H**) *holds. If*

$$0 < \int_0^1 t(1-t) F_1(t) \mathrm{d}t < +\infty, \tag{2.1}$$

then singular superlinear boundary value problem (1.1) and (1.2) has at least one $C^{2n-2}[0, 1]$ positive solution.

Theorem 2.2. *Suppose* (**H**) *holds with $0 < \mu_n < 1$, then a necessary and sufficient condition for problem (1.1) and (1.2) to have $C^{2n-2}[0, 1]$ positive solutions is that (2.1) holds.*

Theorem 2.3. *Suppose* (**H**) *holds with $0 < \mu_n < 1$, then a necessary and sufficient condition for problem (1.1) and (1.2) to have $C^{2n-1}[0, 1]$ positive solutions is that the following integral conditions hold:*

$$0 < \int_0^1 F_2(t) \mathrm{d}t < +\infty. \tag{2.2}$$

To prove the theorems, we state a fixed-point theorem in a cone as follows.

Lemma 2.1 (*See [19]*). *Let E be a Banach space and P a cone in E. Suppose that Ω_1 and Ω_2 are two bounded open subsets of E with $\theta \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$. If $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator satisfying*

$$\|Tx\| \le \|x\|, \quad \text{for } x \in P \cap \partial\Omega_1 \quad \text{and} \quad \|Tx\| \ge \|x\|, \quad \text{for } x \in P \cap \partial\Omega_2$$

then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Let $E_1 = \{x \in C^{2n-2}[0, 1] \mid x^{(2i)}(0) = x^{(2i)}(1) = 0, 0 \le i \le n-1\}$. For every $x \in E_1$, we define the norm $\|x\|_1 = |x^{(2n-2)}|_0$, where $|\cdot|_0$ is the usual sup-norm for continuous functions on $[0, 1]$. It is clear that E_1 equipped with norm $\|\cdot\|_1$ is a Banach space. We denote by E the Banach space $E = \{x \in C^2[0, 1] \mid x(0) = x(1) = 0\}$ equipped with the norm $\|x\| = |x''|_0$.

Let $G(t,s)$ be the Green's function of the second order boundary value problem

$$-x''(t)=0,\quad 0<t<1,$$
$$x(0)=x(1)=0,$$

that is

$$G(t,s)=\begin{cases}t(1-s), & 0\le t\le s\le 1,\\ s(1-t), & 0\le s\le t\le 1.\end{cases}$$

It is easy to verify that for $t,s\in[0,1]\times[0,1]$,

$$G(t,t)G(s,s)=t(1-t)s(1-s)\le G(t,s)\le t(1-t)(\text{or } s(1-s)). \tag{2.3}$$

If we let $H(t,s)=\int_0^1\cdots\int_0^1 G(t,s_{n-1})\dots G(s_1,s)\mathrm{d}s_1\dots\mathrm{d}s_{n-1}$, then $H(t,s)$ is the Green's function of the $2n$th-order boundary value problem

$$(-1)^n x^{(2n)}(t)=0,\quad 0<t<1,$$
$$x^{(2i)}(0)=x^{(2i)}(1)=0,\quad i=0,1,\dots,n-1.$$

Let $H_0(t,s)=H(t,s)$, $H_{n-1}(t,s)=G(t,s)$, and

$$H_i(t,s)=\int_0^1\cdots\int_0^1 G(t,s_{n-i-1})\dots G(s_1,s)\mathrm{d}s_1\dots\mathrm{d}s_{n-i-1},\quad i=0,1,\dots,n-2.$$

Indeed, for every $i\in[0,n-1]$, H_i is the Green's function of the $2(n-i)$th-order boundary value problem

$$(-1)^{n-i}x^{(2n-2i)}(t)=0,\quad 0<t<1,$$
$$x^{(2j)}(0)=x^{(2j)}(1)=0,\quad j=0,1,\dots,n-i-1.$$

Lemma 2.2. *For $i=0,1,\dots,n-2$, we have*

$$\left(\frac{1}{30}\right)^{n-i-1}t(1-t)s(1-s)\le H_i(t,s)\le t(1-t)s(1-s),\quad \forall\, t,s\in[0,1].$$

Proof. From properties of $G(t,s)$, the fact that $\int_0^1 s^2(1-s)^2\mathrm{d}s=\frac{1}{30}$, for $i=0,1,\dots,n-2$, we have

$$H_i(t,s)\le\int_0^1\cdots\int_0^1 t(1-t)s(1-s)\mathrm{d}s_1\dots\mathrm{d}s_{n-i-1}=t(1-t)s(1-s)$$

and

$$\begin{aligned}H_i(t,s)&\ge\int_0^1\cdots\int_0^1 G(t,t)G(s_{n-i-1},s_{n-i-1})\dots G(s_1,s_1)G(s,s)\mathrm{d}s_1\dots\mathrm{d}s_{n-i-1}\\&\ge\left(\frac{1}{30}\right)^{n-i-1}t(1-t)s(1-s).\end{aligned}$$

The proof of Lemma 2.2 is completed. □

Let $e(t)=t(1-t)$. Set

$$E_e=\{x\in C[0,1]\mid \text{there exists } \lambda>0 \text{ such that } -\lambda e\le x\le\lambda e\},$$

and

$$\|x\|_e=\inf\{\lambda>0\mid -\lambda e\le x\le\lambda e\},\quad \forall\, x\in E_e.$$

It is easy to see that E_e becomes a Banach space under the norm $\|\cdot\|_e$. $\|x\|_e$ is called the e-Norm of the element $x\in E_e$.

Denote

$$P=\{x\in E_1\mid (-1)^i x^{(2i)}(t)\ge 0, (-1)^{n-1}x^{(2n-2)}(t)\ge t(1-t)\|x\|, i=0,1,\dots,n-1\}.$$

It can be easily seen that P is a cone in E_1.

Lemma 2.3. *Suppose $x\in P$. Then for $i=0,1,\dots,n-2$, $x^{(2i)}\in E_e$ and $\left(\frac{1}{30}\right)^{n-i-1}\|x\|\le\|x^{(2i)}\|_e\le\|x\|$.*

Proof. For $x\in P$ and $n\ge 2$, we have $x\in C^{2n-2}[0,1]$, and for $i=0,1,\dots,n-2$,

$$(-1)^i x^{(2i)}(t)=\int_0^1 H_{1+i}(t,s)(-1)^{(n-1)}x^{(2n-2)}(s)\mathrm{d}s\le|x^{(2n-2)}|_0 t(1-t)=\|x\|t(1-t).$$

Thus, $x^{(2i)}\in E_e$ and $\|x^{(2i)}\|_e\le\|x\|$. On the other hand, by Lemma 2.2, we have

$$\begin{aligned}(-1)^i x^{(2i)}(t) &= \int_0^1 H_{1+i}(t,s)(-1)^{(n-1)} x^{(2n-2)}(s)\mathrm{d}s \\ &\geq \left(\frac{1}{30}\right)^{n-i-2} t(1-t)\|x\| \int_0^1 s^2(1-s)^2 \mathrm{d}s \geq \left(\frac{1}{30}\right)^{n-i-1} t(1-t)\|x\|.\end{aligned}$$

The proof of Lemma 2.3 is completed. □

Next, we define an operator $T : P \to E_1$ by

$$(Tx)(t) = \int_0^1 H(t,s) f\left(s, x(s), -x''(s), \ldots, (-1)^{n-1} x^{(2n-2)}(s)\right) \mathrm{d}s, \quad x \in P.$$

For $x \in P$, let c be a positive number such that $\frac{\|x\|}{c} < 1$ and $c > 1$. From (1.3) and (1.5), and Lemma 2.3,

$$\begin{aligned}(Tx)(t) &= \int_0^1 H(t,s) f\left(s, x(s), -x''(s), \ldots, (-1)^{n-1} x^{(2n-2)}(s)\right) \mathrm{d}s \\ &\leq \int_0^1 s(1-s) f\left(s, \frac{cx(s)}{cs(1-s)} s(1-s), \frac{-cx''(s)}{cs(1-s)} s(1-s), \ldots, c\frac{(-1)^{n-1} x^{(2n-2)}(s)}{c}\right) \mathrm{d}s \\ &\leq c^{\sum_{i=1}^n (\mu_i - \lambda_i)} \|x\|^{\sum_{i=1}^n \lambda_i} \int_0^1 s(1-s) F_1(s) \mathrm{d}s.\end{aligned}$$

Hence, if (2.1) in Theorem 2.1 holds, T is well defined on P. Moreover, suppose that f satisfies (2.1), by Lemma 2.3 and Fubini's theorem, we have, for $x \in P$, $i = 0, 1, \ldots, n-2$, that

$$\begin{aligned}&\int_0^1 \cdots \int_0^1 G(t, s_{n-i-1}) \ldots G(s_1, s) f\left(s, x(s), -x''(s), \ldots, (-1)^{n-1} x^{(2n-2)}(s)\right) \mathrm{d}s\mathrm{d}s_1 \ldots \mathrm{d}s_{n-i-1} \\ &\quad = \int_0^1 \cdots \int_0^1 G(t, s_{n-i-1}) \ldots G(s_1, s) f\left(s, x(s), -x''(s), \ldots, (-1)^{n-1} x^{(2n-2)}(s)\right) \mathrm{d}s_1 \ldots \mathrm{d}s_{n-i-1}\mathrm{d}s.\end{aligned}$$

Then, a positive solution to the integral equation $x = Tx$ is a positive solution of the boundary value problem (1.1) and (1.2).

Lemma 2.4. *If (2.1) holds, then $T(P) \subset P$.*

Proof. Let $x \in P$. For $i = 0, 1, \ldots, n-1$, from the definition of $H_i(t,s)$, we have

$$(-1)^i x^{(2i)}(t) = \int_0^1 H_i(t,s) f\left(s, x(s), -x''(s), \ldots, (-1)^{n-1} x^{(2n-2)}(s)\right) \mathrm{d}s,$$

and $(-1)^i x^{(2i)}(t) > 0$. By (2.3), we have that

$$\begin{aligned}(-1)^{n-1} x^{(2n-2)}(\tau) &= \int_0^1 H_{n-1}(\tau,s) f\left(s, x(s), -x''(s), \ldots, (-1)^{n-1} x^{(2n-2)}(s)\right) \mathrm{d}s \\ &\leq \int_0^1 s(1-s) f\left(s, x(s), -x''(s), \ldots, (-1)^{n-1} x^{(2n-2)}(s)\right) \mathrm{d}s, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned}(-1)^{n-1} x^{(2n-2)}(t) &= \int_0^1 H_{n-1}(t,s) f\left(s, x(s), -x''(s), \ldots, (-1)^{n-1} x^{(2n-2)}(s)\right) \mathrm{d}s \\ &\geq t(1-t) \int_0^1 s(1-s) f\left(s, x(s), -x''(s), \ldots, (-1)^{n-1} x^{(2n-2)}(s)\right) \mathrm{d}s. \end{aligned} \tag{2.5}$$

Using (2.4) and (2.5), we have

$$(-1)^{n-1} x^{(2n-2)}(t) \geq t(1-t)(-1)^{n-1} x^{(2n-2)}(\tau), \quad \forall\, t, \tau \in [0,1],$$

then $(-1)^{n-1} x^{(2n-2)}(t) \geq t(1-t)\|x\|$, i.e., $T(P) \subset P$. We complete the proof. □

Lemma 2.5. *If (2.1) holds, then T is a completely continuous operator on P.*

Proof. We show that T is bounded. In fact, let $V \subset P$ be a bounded set. There exists a positive constant L satisfying $\|x\| \leq L$ for all $x \in V$. Let c be a constant such that $\frac{L}{c} < 1$ and $c > 1$. By Lemma 2.3, we get

$$\begin{aligned}\left|(Tx)^{(2n-2)}(t)\right| &= \int_0^1 H(t,s)f\left(s, x(s), -x''(s), \ldots, (-1)^{n-1}x^{(2n-2)}(s)\right) \mathrm{d}s \\ &\leq \int_0^1 s(1-s)f\left(s, \frac{cx(s)}{cs(1-s)}s(1-s), \frac{-cx''(s)}{cs(1-s)}s(1-s), \ldots, c\frac{(-1)^{n-1}x^{(2n-2)}(s)}{c}\right) \mathrm{d}s \\ &\leq C_1 \int_0^1 s(1-s)F_1(s)\mathrm{d}s < \infty, \end{aligned} \tag{2.6}$$

where

$$C_1 = c^{\sum\limits_{i=1}^{n}(\mu_i - \lambda_i)} L^{\sum\limits_{i=1}^{n}\lambda_i}. \tag{2.7}$$

The definition of norm $\|\cdot\|$ together with (2.1) implies

$$\|(Tx)\| \leq C_1 \int_0^1 s(1-s)F_1(s)\mathrm{d}s. \tag{2.8}$$

Namely, $\{TV\}$ is uniformly bounded.

Secondly, by (2.8) and the Ascoli–Arzela theorem, we need to show only that TV is equicontinuous on $[0, 1]$. Since $\{TV\}$ is bounded in P, $\{(-1)^{n-1}(Tx)^{(2n-2)}(t) : x \in V\}$ is bounded. Hence, for $i = 0, 1, \ldots, 2n-3$, $\{(Tx)^{(i)}(t) : x \in V\}$ is equicontinuous. Therefore, we need only to prove that $\{(Tx)^{(2n-2)}(t) : x \in V\}$ is equicontinuous. Let c be a constant such that $\frac{L}{c} < 1$ and $c > 1$. Since

$$\begin{aligned}\left|(Tx)^{(2n-1)}(t)\right| &\leq \int_0^t sf\left(s, x(s), -x''(s), \ldots, (-1)^{n-1}x^{(2n-2)}(s)\right) \mathrm{d}s \\ &\quad + \int_t^1 (1-s)f\left(s, x(s), -x''(s), \ldots, (-1)^{n-1}x^{(2n-2)}(s)\right) \mathrm{d}s \\ &\leq C_1 \int_0^t sF_1(s)\mathrm{d}s + C_1 \int_t^1 (1-s)F_1(s)\mathrm{d}s =: C_1K(t), \end{aligned}$$

where C_1 is defined by (2.7), we have $\left|[(Tx)^{(2n-2)}(t)]'\right| \leq C_1K(t)$ for any $x \in V$. Now the equicontinuous of $\{(Tx)^{(2n-2)}(t) : x \in V\}$ follows from

$$\begin{aligned}\int_0^1 K(t)\mathrm{d}t &= \int_0^1 \int_0^t sF_1(s)\mathrm{d}s\mathrm{d}t + \int_0^1 \int_t^1 (1-s)F_1(s)\mathrm{d}s\mathrm{d}t \\ &\leq \int_0^1 s(1-s)F_1(s)\mathrm{d}s < +\infty. \end{aligned}$$

Therefore, $\{TV\}$ is relatively compact.

Finally, it remains to show T is continuous. Suppose $x_m, x_0 \in P$, and $\|x_m - x_0\| \to 0$ $(m \to \infty)$. Then $\{x_m\}$ is a bounded set and for $i = 0, 1, \ldots, n-1$,

$$|x_m^{(2i)} - x_0^{(2i)}|_0 \to 0, \ (m \to \infty).$$

Let $L = \sup\{\|x_m\|, m = 0, 1, 2, \ldots\}$. Then we may still choose positive constants c such that $\frac{L}{c} < 1$ and $c > 1$. Similar to the proof of (2.6), we get

$$f\left(t, x(t), -x''(t), \ldots, (-1)^{n-1}x^{(2n-2)}(t)\right) \leq C_1F_1(t), \tag{2.9}$$

and

$$\begin{aligned}|(Tx_m)^{(2n-2)}(t) - (Tx_0)^{(2n-2)}(t)| \leq \int_0^1 s(1-s) \Big| &f\left(s, x_m(s), -x_m''(s), \ldots, (-1)^{n-1}x_m^{(2n-2)}(s)\right) \\ &- f\left(s, x_0(s), -x_0''(s), \ldots, (-1)^{n-1}x_0^{(2n-2)}(s)\right)\Big| \mathrm{d}s. \end{aligned}$$

The above inequality, (2.9), the Lebesgue dominated convergence theorem, and Ascoli–Arzela theorem guarantee that

$$\|Tx_m - Tx_0\| \to 0 \quad (m \to \infty),$$

that is, T is continuous. Summing up, $T : P \to P$ is completely continuous. □

Proof of Theorem 2.1. For $0 < r < 1 < R$, let

$$P_r = \{x \in P : \|x\| \le r\}, \qquad P_R = \{x \in P : \|x\| \le R\}.$$

Choose r such that

$$0 < r \le \min\left\{\left(\int_0^1 s(1-s)F_1(s)\mathrm{d}s\right)^{-\frac{1}{\sum_{i=1}^{n}\lambda_i - 1}}, \frac{1}{2}\right\}.$$

Then for $x \in \partial P_r$, we have

$$\frac{r}{30^{n-i-1}}t(1-t) \le (-1)^i x^{(2i)}(t) \le rt(1-t), \quad \text{for } i = 0, 1, \ldots, n-2,$$
$$rt(1-t) \le (-1)^{n-1}x^{(2n-2)}(t) \le r.$$

By the properties of $G(t,s)$, (2.1), (1.3) and (1.5), we get

$$\begin{aligned}(-1)^{n-1}(Tx)^{(2n-2)}(t) &= \int_0^1 G(t,s)f\left(s, x(s), -x''(s), \ldots, (-1)^{n-1}x^{(2n-2)}(s)\right)\mathrm{d}s \\ &\le \int_0^1 s(1-s)f\,(s, rs(1-s), rs(1-s), \ldots rs(1-s), r)\,\mathrm{d}s \\ &\le r^{\sum_{i=1}^{n}\lambda_i}\int_0^1 s(1-s)F_1(s)\mathrm{d}s \le r = \|x\|, \quad \forall\, x \in \partial P_r.\end{aligned}$$

This guarantees

$$\|Tx\| \le \|x\|, \quad \forall\, x \in \partial P_r. \tag{2.10}$$

On the other hand, choose R such that

$$R \ge \max\left\{30^{n-1}, \left[\left(\frac{1}{30^{n-1}}\right)^{\sum_{i=1}^{n-1}\lambda_i}\left(\frac{3}{16}\right)^{\lambda_n+1}\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)F_1(s)\mathrm{d}s\right]^{-\frac{1}{\sum_{i=1}^{n}\lambda_i - 1}}\right\}.$$

Then for $x \in \partial P_R$, we have

$$t(1-t) \le \frac{R}{30^{n-i-1}}t(1-t) \le (-1)^i x^{(2i)}(t) \le Rt(1-t), \quad \text{for } i = 0, 1, \ldots, n-2,$$
$$Rt(1-t) \le (-1)^{n-1}x^{(2n-2)}(t) \le R.$$

Therefore,

$$\begin{aligned}(-1)^{n-1}(Tx)^{(2n-2)}(t) &= \int_0^1 G(t,s)f\left(s, x(s), -x''(s), \ldots, (-1)^{n-1}x^{(2n-2)}(s)\right)\mathrm{d}s \\ &\ge \frac{3}{16}\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)f\left(s, \frac{x(s)}{s(1-s)}s(1-s), \frac{-x''(s)}{s(1-s)}s(1-s), \ldots, \frac{3R}{16}\right)\mathrm{d}s \\ &\ge \frac{3}{16}\left(\frac{1}{30^{n-1}}\right)^{\sum_{i=1}^{n-1}\lambda_i}\left(\frac{3}{16}\right)^{\lambda_n} R^{\sum_{i=1}^{n}\lambda_i}\int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)F_1(s)\mathrm{d}s \\ &\ge R = \|x\|, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \ \forall x \in \partial P_R.\end{aligned}$$

This guarantees

$$\|Tx\| \ge \|x\|, \quad \forall\, x \in \partial P_R. \tag{2.11}$$

By the complete continuity of T, (2.10) and (2.11), and Lemma 2.1, we obtain that T has a fixed point $x_*(t)$ in $\overline{P_R} \setminus P_r$. Consequently, (1.1) and (1.2) has a $C^{2n-2}[0,1]$ positive solution $x_*(t)$ in $\overline{P_R} \setminus P_r$. □

Proof of Theorem 2.2. *Sufficiency*. This result is a direct consequence of Theorem 2.1.

Necessity. Suppose (1.1) and (1.2) has a $C^{2n-2}[0,1]$ positive solution x. Then $x \in C^{2n-2}[0,1] \cap C^{2n}(0,1)$. For $i = 0, 1, \ldots, n-1$, we have

$$(-1)^i x^{(2i)}(t) > 0, \quad t \in (0,1). \tag{2.12}$$

Similar to the proof of Lemma 2.1 [17], we can get

$$(-1)^{n-1} x^{(2n-2)}(t) = \int_0^1 G(t,s) f\left(s, x(s), -x''(s), \ldots, (-1)^{n-1} x^{(2n-2)}(s)\right) \mathrm{d}s.$$

This means that $\int_0^1 G(t,s) f\left(s, x(s), -x''(s), \ldots, (-1)^{n-1} x^{(2n-2)}(s)\right) \mathrm{d}s$ is well defined on $[0,1]$. This together with (2.3) implies

$$0 < \int_0^1 s(1-s) f\left(s, x(s), -x''(s), \ldots, (-1)^{n-1} x^{(2n-2)}(s)\right) \mathrm{d}s < +\infty$$

and

$$(-1)^{n-1} x^{(2n-2)}(t) \geq t(1-t)(-1)^{n-1}(x)^{(2n-2)}(\tau), \quad \forall\, t, \tau \in [0,1].$$

Then $x \in P$. By Lemma 2.3, we have

$$\left(\frac{1}{30}\right)^{n-i-1} \|x\| \leq \|x^{(2i)}\|_e \leq \|x\|, \quad i = 0, 1, \ldots, n-2.$$

Let c be a constant such that $\frac{c}{\|x\|} > 1$ and $\frac{1}{c} < 1$. Then

$$\begin{aligned}
F_1(t) &= f(t, t(1-t), t(1-t), \ldots, t(1-t), 1) \\
&= f\left(t, \frac{t(1-t)}{x(t)} x(t), \frac{t(1-t)}{-x''(t)}(-x''(t))\right), \ldots, \frac{1}{(-1)^{n-1} x^{(2n-2)}(t)} (-1)^{n-1} x^{(2n-2)}(t) \\
&\leq f\left(t, \frac{1}{c}\frac{c30^{n-1}}{\|x\|} x(t), \frac{1}{c}\frac{c30^{n-1}}{\|x\|}(-x''(t)), \ldots, \frac{1}{c}\frac{c}{(-1)^{n-1} x^{(2n-2)}(t)} (-1)^{n-1} x^{(2n-2)}(t)\right) \\
&\leq C_0 |x^{(2n-2)}(t)|^{-\mu_n} f(t, x(t), -x''(t), \ldots, (-1)^i x^{(2i)}(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t)),
\end{aligned}$$

where $C_0 = c^{\sum_{i=1}^{n}(\mu_i - \lambda_i)} \left(\frac{30^{n-1}}{\|x\|}\right)^{\sum_{i=1}^{n-1} \mu_i}$. That is

$$F_1(t) |x^{(2n-2)}(t)|^{\mu_n} \leq C_0 f(t, x(t), -x''(t), \ldots, (-1)^i x^{(2i)}(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t)). \tag{2.13}$$

Since $x^{(2n-2)}(0) = x^{(2n-2)}(1) = 0$, there must exists a $t_0 \in (0,1)$ such that $x^{(2n-1)}(t_0) = 0$. Then $x^{(2n-1)}(t) = -\int_t^{t_0} x^{(2n)}(s)\mathrm{d}s$ for $t \in (0,1)$. It is easy to see that, for $0 < t < t_0$, $(-1)^{n-1} x^{(2n-2)}(t)$ is nondecreasing. Recalling the definition of $F_1(t)$, from (2.13), we have

$$\begin{aligned}
|x^{(2n-2)}(t)|^{\mu_n} \int_t^{t_0} F_1(s)\mathrm{d}s &\leq \int_t^{t_0} |x^{(2n-2)}(s)|^{\mu_n} F_1(s)\mathrm{d}s \\
&\leq C_0 \int_t^{t_0} f(s, x(s), -x''(s), \ldots, (-1)^i x^{(2i)}(s), \ldots, (-1)^{n-1} x^{(2n-2)}(s))\mathrm{d}s \\
&= C_0 \int_t^{t_0} (-1)^n x^{(2n)}(s)\mathrm{d}s = C_0 (-1)^{n-1} x^{(2n-1)}(t).
\end{aligned}$$

Hence, we have

$$\int_t^{t_0} f(s, s(1-s), s(1-s), \ldots, s(1-s), 1)\mathrm{d}s \leq C_0 (-1)^{n-1} x^{(2n-1)}(t)[(-1)^{n-1} x^{(2n-2)}(t)]^{-\mu_n}. \tag{2.14}$$

From (2.14),

$$\begin{aligned}
\int_0^{t_0} s f(s, s(1-s), s(1-s), \ldots, s(1-s), 1)\mathrm{d}s &= \int_0^{t_0}\int_t^{t_0} f(s, s(1-s), s(1-s), \ldots, s(1-s), 1)\mathrm{d}s\mathrm{d}t \\
&\leq C_0 \int_0^{t_0} (-1)^{n-1} x^{(2n-1)}(t)[(-1)^{n-1} x^{(2n-2)}(t)]^{-\mu_n}\mathrm{d}t \\
&= C_0 (1-\mu_n)^{-1} (-1)^{n-1} [x^{(2n-2)}(t)]^{1-\mu_n} < +\infty.
\end{aligned}$$

We can also, similarly, get $\int_{t_0}^{1}(1-s)f(s, s(1-s), s(1-s), \ldots, s(1-s), 1)\mathrm{d}s < +\infty$. Hence, $\int_0^1 s(1-s)f(s, s(1-s), s(1-s), \ldots, s(1-s), 1)\mathrm{d}s < +\infty$. Then we have $\int_0^1 s(1-s)F_1(t)\mathrm{d}s < +\infty$. We complete the proof of the necessity of Theorem 2.2. $\square$

Proof of Theorem 2.3. *Sufficiency*. Assume (2.2) holds. According to (1.3) and (1.5), we have

$$t(1-t)F_1(t) \le [t(1-t)]^{\mu_n}F_1(t) \le F_2(t) \le F_1(t), \quad t \in (0,1).$$

This implies (2.1) holds. We define a set $Q \subset C^{2n-2}[0,1]$ by

$$Q = \{x \in P : \exists\, c_x > 0,\ 0 \le (-1)^{n-1}x^{(2n-2)}(t) \le c_x t(1-t),\ t \in [0,1]\}.$$

It is easy to verify that Q is a cone and $Q \subset P$. In the following, we prove that $T : Q \to Q$ is well defined. For $x \in Q$, there exist $c_x \ge 1$ such that $0 \le (-1)^{n-1}x^{(2n-2)}(t) \le c_x t(1-t)$. By the definition of Q and Lemma 2.4, we only need to to prove that there exists a constant c_{Tx} such that $(-1)^{n-1}(Tx)^{(2n-2)}(t) \le c_{Tx}t(1-t)$. Let c be a constant such that $\frac{\|x\|}{c} < 1$ and $c > 1$. By (1.3), we get

$$\begin{aligned}(-1)^{n-1}(Tx)^{(2n-2)}(t) &= \int_0^1 G(t,s)f\left(s, x(s), -x''(s), \ldots, (-1)^{n-1}x^{(2n-2)}(s)\right)\mathrm{d}s \\ &\le \int_0^1 G(t,s)f\,(s, \|x\|s(1-s), \|x\|s(1-s), \ldots \|x\|s(1-s), c_x s(1-s))\,\mathrm{d}s \\ &\le c^{\sum\limits_{i=1}^{n-1}(\mu_i-\lambda_i)}\|x\|^{\sum\limits_{i=1}^{n-1}\lambda_i}(c_x)^{\mu_n}t(1-t)\int_0^1 F_2(s)\mathrm{d}s. \end{aligned} \tag{2.15}$$

Let $c_{Tx} = c^{\sum_{i=1}^{n-1}(\mu_i-\lambda_i)}\|x\|^{\sum_{i=1}^{n-1}\lambda_i}(c_x)^{\mu_n}\int_0^1 F_2(s)\mathrm{d}s$. By (2.2), we know $c_{Tx} > 0$, so $(-1)^{n-1}(Tx)^{(2n-2)}(t) \le c_{Tx}t(1-t)$. Similar to the proof of Theorem 2.1, we can get

$$\|Tx\| \ge \|x\|, \quad \forall\, x \in \partial Q_R; \qquad \|Tx\| \le \|x\|, \quad \forall\, x \in \partial Q_r, \tag{2.16}$$

where R, r are given as in Theorem 2.1. By the complete continuity of T, (2.16), and Lemma 2.1, we obtain that T has a fixed point $x_*(t)$ in $\overline{Q_R} \setminus Q_r$. Consequently, (1.1) and (1.2) has a $C^{2n-2}[0,1]$ positive solution $x_*(t)$ in $\overline{Q_R} \setminus Q_r$. Let c be a constant such that $\frac{\|x_*\|}{c} < 1$ and $c > 1$. Similar to the proof of (2.15), we get

$$|x_*^{(2n)}(t)| \le f\left(t, x_*(t), -x_*''(t), \ldots, (-1)^{n-1}x_*^{(2n-2)}(t)\right) \le c^{\sum\limits_{i=1}^{n}(\mu_i-\lambda_i)}\|x_*\|^{\sum\limits_{i=1}^{n-1}\lambda_i}(c_{x_*})^{\mu_n}F_2(t).$$

Hence $x_*^{(2n)}(t)$ is absolutely integrable on $[0,1]$. This implies $x_*(t) \in C^{2n-1}[0,1]$, so $x_*(t)$ is a $C^{2n-1}[0,1]$ positive solution of the problem (1.1) and (1.2).

Necessity. Suppose (1.1) and (1.2) has a $C^{2n-1}[0,1]$ positive solution x. Then both $(-1)^{2n-1}x^{(2n-1)}(0) > 0$ and $(-1)^{2n-1}x^{(2n-1)}(1) < 0$ exist. By Lemma 2.3 and the proof of Theorem 2.2, we have

$$\left(\frac{1}{30}\right)^{n-i-1}\|x\| \le \|x^{(2i)}\|_e \le \|x\|, \quad i = 0, 1, \ldots, n-2,$$

and

$$(-1)^{n-1}x^{(2n-2)}(t) \ge t(1-t)\|x\|.$$

Let c be a constant such that $\frac{c}{\|x\|} > 1$ and $\frac{1}{c} < 1$. Then

$$\begin{aligned}F_2(t) &= f(t, t(1-t), t(1-t), \ldots, t(1-t), t(1-t)) \\ &= f\left(t, \frac{t(1-t)}{x(t)}x(t), \frac{t(1-t)}{-x''(t)}(-x''(t)), \ldots, \frac{t(1-t)}{(-1)^{n-1}x^{(2n-2)}(t)}(-1)^{n-1}x^{(2n-2)}(t)\right) \\ &\le f\left(t, \frac{1}{c}\frac{c30^{n-1}}{\|x\|}x(t), \frac{1}{c}\frac{c30^{n-1}}{\|x\|}(-x''(t)), \ldots, \frac{1}{c}\frac{c}{\|x\|}(-1)^{n-1}x^{(2n-2)}(t)\right) \\ &\le C_0 f(t, x(t), -x''(t), \ldots, (-1)^i x^{(2i)}(t), \ldots, (-1)^{n-1}x^{(2n-2)}(t)),\end{aligned}$$

where $C_0 = c^{\sum_{i=1}^{n}(\mu_i-\lambda_i)}(\frac{30^{n-1}}{\|x\|})^{\sum_{i=1}^{n-1}\mu_i}(\frac{1}{\|x\|})^{\mu_n}$. That is

$$F_2(t) \le C_0 f(t, x(t), -x''(t), \ldots, (-1)^i x^{(2i)}(t), \ldots, (-1)^{n-1}x^{(2n-2)}(t)).$$

Consequently,

$$\begin{aligned}\int_0^1 F_2(t)\mathrm{d}t &\le C_0\int_0^1 f(t, x(t), -x''(t), \ldots, (-1)^i x^{(2i)}(t), \ldots, (-1)^{n-1}x^{(2n-2)}(t))\\ &= C_0((-1)^{2n-1}x^{(2n-1)}(0) - (-1)^{2n-1}x^{(2n-1)}(1)) < +\infty.\end{aligned}$$

Thus, (2.2) holds. □

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